Observability Problem of Some Stochastic Partial Differential Equations

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Outline

1. Introduction

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- Roughly speaking, it concerns whether one can recover the state of a system by some partial knowledge of the state (which is called the observation of the system).
- For an equation, it means that whether one can determine the solution uniquely by some partial knowledge of the equation (this is called the unique continuation problem for the equation).

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- If u vanishes infinite order at a point $x \in G$, can we conclude that $u|_G = 0$?
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- The unique continuation prperty is still true. This can be proved by T. Carleman in 1939.

- Let us recall Carleman's result briefly. We begin with the following notion:

  A function $y \in L^2_{loc}(\mathbb{R}^n)$ *is said to vanish of infinite order at* $x_0 \in \mathbb{R}^n$ *if there exists an* $R > 0$ *so that for each integer* $N \in \mathbb{N}$*, there is a constant* $C_N > 0$ *satisfying that*

$$\int_{\mathcal{B}(x_0,r)} y^2 dx \leq C_N r^{2N}, \qquad \forall\, r \in (0, R).$$

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- Let $P = -\Delta + V$ with $V \in L^\infty_{loc}(\mathbb{R}^2)$. T. Carleman showed that *any solution $y \in H^1_{loc}(\mathbb{R}^2)$ to $Py = 0$ (in the sense of distribution) equals zero if it vanishes of infinite order at some $x_0 \in \mathbb{R}^2$*. To prove this result, he introduced a new method, now known as the Carleman estimate.

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- In 1954, C. Müller extended the above method to elliptic equations on $\mathbb{R}^n$.

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- Generally speaking, one cannot get the above strong result. For example, due to the finite speed of propagation, the about unique continuation property does not hold for hyperbolic equation.
- Some weaker formulations are as follows:
- If u vanishes in a subset $F \subset G$, can we conclude that $u|_G = 0$?
- Does $u|_F = 0$ imply $u|_{O(F)} = 0$? Here $O(F)$ is an (open) neighborhood of F.

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- Classical results/tools include Carleman estimate, Frequency method, and so on.

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- Some typical applications are as follows:
- In Control theory, an approximate controllability problem can be reduced to a suitable unique continuation problem (e.g. Russell, SIAM Rev. 20 (1978)).
- In Inverse problems, the uniqueness of the unknown coefficients can be reduced to a suitable unique continuation problem (e.g. Klibanov, Inverse Problems 8 (1992)).

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- Bourgain & Kenig, Invent. Math. 2005, for the study of the Anderson localization.
- Escauriaza, Kenig, Ponce & Vega, Comm. Math. Phys. 2011, for the study of concentration profiles of blow-up solutions.

2. Carleman estimate for PDEs

- Let $P(x,D)$ be a m-th order partial differential operator with smooth bounded coefficients. A Carleman estimate is an estimate in the following forms:

$$\sum_{|\alpha|\le m-1} \tau^{2(m-|\alpha|)-1} \int_{\mathbb{R}^n} |D^\alpha u|^2 \, e^{2\tau\varphi} dx$$

$$\le K_1 \int_{\mathbb{R}^n} |P(x,D)u|^2 e^{2\tau\varphi} dx, \quad u \in C_0^\infty(G), \ \tau > \tau_0;$$

or

$$\sum_{|\alpha|\le m-1} \tau^{2(m-|\alpha|)-1} \int_{\mathbb{R}^n} |D^\alpha u|^2 \, e^{2\tau\varphi} dx$$

$$\le K_2 \int_{\mathbb{R}^n} |P(x,D)u|^2 e^{2\tau\varphi} dx + K_3 \sum_{|\alpha|\le m-2} \tau^{2(m-|\alpha|)-1} \int_{\mathbb{R}^n} |D^\alpha u|^2 \, e^{2\tau\varphi} dx,$$

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- Let $a > 0$. It holds that

$$2a \int_{\mathbb{R}} |u|^2 e^{at^2} dt \leq \int_{\mathbb{R}} \left| \frac{du}{dt} \right|^2 e^{at^2} dt, \quad u \in C_0^\infty(\mathbb{R}). \qquad (3)$$

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- *Proof.* Set $v(t) = u(t)e^{at^2/2}$. By means of an integration by parts,

$$\begin{aligned} \int_{\mathbb{R}} \left| u'(t) \right|^2 e^{at^2} dt &= \int_{\mathbb{R}} \left| v'(t) - atv(t) \right|^2 dt \\ &= \int_{\mathbb{R}} \left| v'(t) + atv(t) \right|^2 dt + 2a \int_{\mathbb{R}} |v(t)|^2 dt \\ &\ge 2a \int_{\mathbb{R}} |u(t)|^2 e^{at^2} dt. \end{aligned}$$

- Let α be a real constant. The estimate holds

$$4\alpha \int_{\mathbb{R}^2} |v|^2 e^{\alpha\left(t^2+s^2\right)} dsdt \leq \int_{\mathbb{R}^2} \left|\frac{\partial v}{\partial s} + i\frac{\partial v}{\partial t}\right|^2 e^{\alpha\left(t^2+s^2\right)} dsdt.$$

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- Writing

$$v(s,t)e^{\frac{1}{2}\alpha\left(s^2+t^2\right)} = w(s,t).$$

By integration by parts, we have

$$\begin{aligned}&\int_{\mathbb{R}^2}\left|\frac{\partial v}{\partial s}+i\frac{\partial v}{\partial t}\right|^2 e^{\alpha\left(t^2+s^2\right)}dsdt\\ &=\int_{\mathbb{R}^2}\left|\frac{\partial w}{\partial s}+i\frac{\partial w}{\partial t}-\alpha(s+it)w\right|^2 dsdt\\ &=\int_{\mathbb{R}^2}\left|\frac{\partial w}{\partial s}-i\frac{\partial w}{\partial t}+\alpha(s-it)w\right|^2 dsdt+4\alpha\int_{\mathbb{R}^2}|w|^2 dsdt.\end{aligned}$$

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where $a_{jk} = a_{kj}$, $j, k = 1, \cdots, n$.

- Let $b = (b_1, \cdots, b_n)$ be a vector in C^n, then it holds that

$$2 \sum_{j,k=1}^{n} a_{jk} b_j \bar{b}_k \int_{\mathbb{R}^n} |u|^2 e^A dx \leq \int_{\mathbb{R}^n} \Big| \sum_{j=1}^{n} b_j D_j u \Big|^2 e^A dx, \ \forall u \in C_0^{\infty}(R_n).$$

3. Observability of Stochastic Heat Equations

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$$dy - \Delta y dt = aydt + bydW(t) \qquad \text{in } G \times (0,T). \tag{4}$$

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- **Theorem 4**(X. Zhang, Differential Integral Equations,2008): Let $G_0 \subset G$. If $y = 0$ in $G_0 \times (0, T)$, $\mathbb{P}$-a.s., then $y = 0$ in $G \times (0, T)$, $\mathbb{P}$-a.s.

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- **Theorem 5**(S. Tang, et al, SICON, 2009): Let $G_0 \subset G$. If G be bounded domain with a C^2 boundary, $y(0) \in L^2(G)$ and $y = 0$ on $(0, T) \times \partial G$, then for any $t \in (0, T]$,

$$\mathbb{E}|y(t)|^2_{L^2(G)} \leq C(t) \int_0^T \int_{G_0} |y|^2 dxds.$$

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- **Theorem 6**(L & Z. Yin, ESAIM:COCV,2015): Let $G_0 \subset G$.
  1. If $y = 0$ on $G_0 \times \{t_0\}$, $\mathbb{P}$-a.s., for a $t_0 \in (0, T]$, then $y = 0$ on $G \times \{t_0\}$, $\mathbb{P}$-a.s.
  2. If $y = 0$ on $\partial G \times (0, T)$, then $y = 0$ on $G \times (0, T)$, $\mathbb{P}$-a.s.
  3. If G is bounded and convex, then

$$\mathbb{E}|y(T_0)|^2_{L^2(G)} \leq C|y(0)|^{2-2\delta}_{L^2(G)}\big(\mathbb{E}|y(T_0)|^2_{L^2(G_0)}\big)^{\delta}$$

for some $\delta \in (0,1)$.

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- **Theorem 7**(QL,2020): Let $y \in L^2_{\mathbb{F}}(0,T;H^1_{loc}(\Omega)) \cap L^2_{\mathbb{F}}(\Omega; C([0,T];L^2_{loc}(B_1))$ solves

$$dy - \Delta y dt = aydt + bydW \text{ in } \Omega \times (0,T). \tag{5}$$

If for every $k \in \mathbb{N}$ we have

$$\mathbb{E}\int_{B_r} y^2(x,t_0)\,dx = O\big(r^{2k}\big), \text{ as } r \to 0, \tag{6}$$

then $y(\cdot,t_0) = 0$, in G, $\mathbb{P}$-a.s. Furthermore, if (8) holds for any $t \in (0,T)$, then $y = 0$, in $G \times (0,T)$, $\mathbb{P}$-a.s.

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- Then,
$$dz = \Delta z dt - (b_t W - a + b^2 + W\Delta b + |\nabla b|^2 W^2) z dt + 2W\nabla b\cdot\nabla z dt.$$
- Hence, z solves the following heat equation with random coefficients
$$z_t - \Delta z = 2e^{\ell} W\nabla b\cdot\nabla z - (b_t W - a + b^2 + W\Delta b + |\nabla b|^2 W^2) z. \tag{7}$$

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- If for every $k \in \mathbb{N}$ we have

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then $y(\cdot, t_0) = 0$, in G, $\mathbb{P}$-a.s. Furthermore, if $y = 0$ on $\partial G \times (0, T)$, then $y = 0$, in $G \times (0, T)$, $\mathbb{P}$-a.s.

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- However, by the above argument, we need a pointwise assumption rather than the assumption on the expectation.

- Theorem 7 is a corollary of following result:

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- **Theorem 8(two-sphere one-cylinder inequality in the interior)**(QL, 2016): Let R be a positive number and $t_0 \in \mathbb{R}$. Assume that $u \in L^2_{\mathbb{F}}(t_0 - R^2, t_0; H^1(B_1)) \cap L^2_{\mathbb{F}}(\Omega; C([t_0 - R^2, t_0]; L^2(B_1))$ is a solution to

$$du - \Delta u dt = a u dt + b u dW \text{ in } B_R \times \left(t_0 - R^2, t_0\right]. \tag{9}$$

Then there exist constants $\eta_1 \in (0,1)$ and C, $C \geq 1$, such that for every r and ρ such that $0 < r \leq \rho \leq \eta_1 R$ we have

$$\mathbb{E}\int_{B_\rho} u^2(x,t_0)dx \leq \frac{CR}{\rho}\left(R^{-2}\mathbb{E}\int_{B_R\times(t_0-R^2,t_0)} u^2 dxdt\right)^{1-\theta_1}\left(\mathbb{E}\int_{B_r} u^2(x,t_0)dx\right)^{\theta_1}, \tag{10}$$

where

$$\theta_1 = \frac{1}{C\log\frac{R}{r}}. \tag{11}$$

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- Let $\gamma \geq 1$ and

$$\sigma(s) = s\exp\left(-\int_0^{\gamma s}\left(1-\exp\left(-\int_0^t \frac{\theta(\eta)}{\eta}d\eta\right)\right)\frac{dt}{t}\right). \tag{13}$$

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- Set

$$\phi(x,t) = -\frac{|x|^2}{8(T_0-t+\lambda)} - (\alpha+1)\log\sigma(T_0-t+\lambda). \tag{14}$$

- **Lemma 1**: There exist constants $C \geq 1$, $\eta_0 \in (0,1)$ and $\delta_1 \in (0,1)$, such that for every α, $\alpha \geq 2$, λ, $0 < \lambda \leq \dfrac{\delta^2}{4\alpha}$, $\delta \in (0, \delta_1]$ and u solves

$$du - \Delta u dt = audt + budW(t) \quad \text{in } \mathbb{R}^n \times (0, \infty),$$

with supp $u \subset Q \triangleq B_{\eta_0} \times [0, \delta^2/2\alpha)$, the following inequality holds true

$$\mathbb{E}\int_Q \Big\{(T_0-t+\lambda)\big[\Delta v+(|\nabla\phi|^2-\partial_t\phi)v\big]-\frac{1}{2}v\Big\}(T_0-t+\lambda)e^{t\phi}(du-\Delta udt)\,dx$$

$$+C\left(e^{C_0}\gamma\right)^{2\alpha+\frac{5}{2}}\mathbb{E}\int_Q \left[u^2+(T_0-t+\lambda)|\nabla u|^2\right]e^{2\phi}dxdt$$

$$\geq \frac{\alpha+1}{C}\mathbb{E}\int_Q \theta(\gamma(T_0-t+\lambda))u^2e^{2\phi}dxdt+\frac{1}{C}\mathbb{E}\int_Q \theta(\gamma(T_0-t+\lambda))t|\nabla u|^2e^{2\phi}dxdt$$

$$+\mathbb{E}\int_Q |Sv|^2dxdt+\lambda^2\mathbb{E}\int_{\mathbb{R}^n}|\nabla v(x,0)|^2dx+\frac{\lambda}{2}\mathbb{E}\int_{\mathbb{R}^n}v(x,0)^2dx$$

$$-\lambda^2\mathbb{E}\int_{\mathbb{R}^n}\left(|\nabla\phi(x,0)|^2-\partial_t\phi(x,0)\right)v^2(x,0)dx,$$

where $Sv = (T_0 - t + \lambda)\big[\Delta v + (|\nabla\phi|^2 + \partial_t\phi)v\big] - \frac{1}{2}v$.

4. Observability for Stochastic Wave Equations

- Consider the following stochastic wave equation:

$$adz_t - \Delta z dt = \left[b_1 z_t + (b_2, \nabla z) + b_3 z\right] dt + b_4 z dW(t) \qquad (15)$$

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- **Theorem 13** (L & Yin, 2020, COCV): Let $x_0 \in \Gamma \setminus \partial\Gamma$ such that $\frac{\partial a(x_0,0)}{\partial \nu} < 0$, and let Γ satisfy the outer paraboloid condition with

$$\kappa < \frac{-\frac{\partial a}{\partial \nu}(x_0, 0)}{4(|a|_{L^\infty(B_\rho(x_0,0))} + 1)}. \qquad (16)$$

Let $y \in L^2_{\mathbb{F}}(\Omega; C([0,2T]; H^1_{loc}(\mathbb{R}^n))) \cap L^2_{\mathbb{F}}(\Omega; C^1([0,2T]; L^2_{loc}(\mathbb{R}^n)))$ solve the equation (1) satisfying that

$$y = \frac{\partial y}{\partial \nu} = 0 \quad \text{on } (0, 2T) \times \Gamma, \ \mathbb{P}\text{-}a.s. \qquad (17)$$

Then, there is a neighborhood $\mathcal{V}$ of x_0 and $T_1 \in (0, T)$ such that

$$y = 0 \quad \text{in } (\mathcal{V} \cap D^+) \times (T - T_1, T + T_1), \ \mathbb{P}\text{-}a.s. \qquad (18)$$

- **Lemma 2**: Let $\ell, \Psi \in C^2((0,T)\times\mathbb{R}^n)$. Assume u is an $H^2_{loc}(\mathbb{R}^n)$-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted process such that u_t is an $L^2(\mathbb{R}^n)$-valued semimartingale. Set $\theta = e^\ell$ and $v = \theta u$. Then, for a.e. $x \in \mathbb{R}^n$ and $\mathbb{P}$-a.s. $\omega \in \Omega$,

$$
\begin{aligned}
&\theta\big(-2a\ell_t v_t + 2\nabla\ell\cdot\nabla v + \Psi v\big)\big(adu_t - \Delta u dt\big)\\
&+\sum_{i=1}^n\Big[\sum_{j=1}^n\big(2\ell_j v_i v_j - \ell_i v_j^2\big) - 2\ell_t v_i v_t + a\ell_i v_t^2 + \Psi v_i v - \Big(A\ell_i + \frac{\Psi_i}{2}\Big)v^2\Big]_i dt\\
&+d\Big[a\sum_{i=1}^n \ell_t v_i^2 - 2a\sum_{i=1}^n \ell_i v_i v_t + a^2\ell_t v_t^2 - a\Psi v_t v + \Big(aA\ell_t + \frac{(a\Psi)_t}{2}\Big)v^2\Big]\\
=&\Big\{\Big[(a^2\ell_t)_t + \sum_{i=1}^n (a\ell_i)_i - a\Psi\Big]v_t^2 - 2\sum_{i=1}^n[(a\ell_i)_t + (a\ell_t)_i]v_i v_t\\
&+\sum_{i=1}^n\Big[(a\ell_t)_t + \sum_{j=1}^n(2\ell_{ij} - \ell_{jj}) + \Psi\Big]v_i^2\\
&+Bv^2 + \big(-2a\ell_t v_t + 2\nabla\ell\cdot\nabla v + \Psi v\big)^2\Big\}dt + a^2\theta^2 l_t (du_t)^2,
\end{aligned}
$$

where A and B are

$$
\begin{cases}
A \triangleq a(\ell_t^2 - \ell_{tt}) - |\nabla\ell|^2 + \Delta\ell - \Psi,\\
B \triangleq A\Psi + (aA\ell_t)_t - \mathrm{div}\,(A\nabla\ell) + \big[(a\Psi)_{tt} - \Delta\Psi\big]/2.
\end{cases}
$$

- Near 0, we will parameterize Γ by

$$x_1 = \gamma(x_2, \cdots, x_n), \ |x_2|^2 + \cdots + |x_n|^2 < \rho. \tag{19}$$

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$$\begin{cases} \kappa < \dfrac{\alpha_0}{4(|a|_{L^\infty(B_\rho(0,0))} + 1)}, \\ -\kappa \displaystyle\sum_{j=2}^{n} |x_j|^2 < \gamma(x_2, \cdots, x_n) \ \text{ if } \displaystyle\sum_{j=2}^{n} |x_j|^2 < \rho. \end{cases} \tag{20}$$

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- Let N satisfy that $1 - 2N\kappa > 0$ and $\alpha N - 2(M_0 + 1) > 0$. We define a weight function by

$$\psi(x,t) = Nx_1 + \frac{1}{2}\sum_{j=1}^{N} |x_j|^2 + \frac{1}{2}t^2 + \frac{1}{2}\varepsilon, \quad \ell = \lambda\psi. \tag{21}$$

- Consider the following equation:

$$\begin{cases} dz_t - \Delta z dt = [b_1 z_t + (b_2, \nabla z) + b_3 z]dt \\ \qquad\qquad\qquad\qquad + b_4 z dW(t) \quad \text{in } G \times (0, T), \\ z = 0 \qquad\qquad \text{on } \partial G \times (0, T). \end{cases} \tag{22}$$

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- **Theorem 14**(X. Zhang, SIMA, 2009): If $z = 0$ in $O_\delta(\Gamma_0) \times (0, T)$, $\mathbb{P}$-a.s., then $z = 0$ in $G \times (0, T)$, $\mathbb{P}$-a.s.

- Consider the following stochastic wave equation:

$$\begin{cases} dz_t - \Delta z dt = (b_1 z_t + b_2 \cdot \nabla z + b_3 z)\, dt \\ \qquad\qquad\qquad + (b_4 z + g) dW(t) & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0) = z_0, \;\; z_t(0) = z_1 & \text{in } G. \end{cases} \tag{23}$$

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- Here b_i $(1 \leq i \leq 4)$ are some suitable known functions; while $(z_0, z_1) \in L^2_{\mathcal{F}_0}(\Omega; H^1_0(G) \times L^2(G))$ and $g \in L^2_{\mathbb{F}}(0, T; L^2(G))$ are unknown.

- **Theorem 15**(L & Zhang, CPAM, 2015): Assume that the solution z to (23) satisfies that $z(T)=0$ in G, $\mathbb{P}$-a.s. Then it holds that

$$
\begin{aligned}
&|(z_0, z_1)|_{L^2_{\mathcal{F}_0}(\Omega;H_0^1(G)\times L^2(G))} + |\sqrt{T-t}g|_{L^2_{\mathbb{F}}(0,T;L^2(G))}\\
&\leq C\left|\frac{\partial z}{\partial \nu}\right|_{L^2_{\mathbb{F}}(0,T;L^2(\Gamma_0))}.
\end{aligned}
$$

- The same conclusion as that in Theorem 10 does NOT hold true even for the deterministic wave equation. Indeed, we choose any $y \in C_0^\infty(Q)$ so that it does not vanish in some proper nonempty subdomain of Q. Put $f = y_{tt} - \Delta y$. Then, it is easy to see that y solves the following wave equation

$$\begin{cases} y_{tt} - \Delta y = f & \text{in } Q, \\ y = 0, & \text{on } \Sigma, \\ y(0) = 0, \ y_t(0) = 0 & \text{in } G. \end{cases}$$

One can show that $y(T) = 0$ in G and $\frac{\partial y}{\partial \nu} = 0$ on Σ. However, it is clear that f does not vanish in Q.

The main difficulty of the study of the control and observation problems for SPDEs.

- 1. Very few are known for SPDEs., i.e., the well-posedness results of the nonhomogeneous boundary value problems for SPDEs, the propagation of singularities results of the solution for SPDEs, etc.

The main difficulty of the study of the control and observation problems for SPDEs.

- 1. Very few are known for SPDEs., i.e., the well-posedness results of the nonhomogeneous boundary value problems for SPDEs, the propagation of singularities results of the solution for SPDEs, etc.
- 2. The stochastic settings lead some useful methods invalid, for example, the lost of the compact embedding for the state spaces, i.e., although $L^2(\Omega; H_0^1(G)) \subset L^2(\Omega; L^2(G))$, the embedding is not compact, which violates the compactness -uniqueness argument. Another example is that the irregularity of the solution with respect to the time variable.

Thank you!